Approximating Symmetric Linear Systems with recursive partial-Hankel Series Expansion

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Abstract. This paper presents an idea of transforming and approximating a symmetric linear system into a partial-Hankel series system. The motivation behind the transformation is to try to enable fast algorithms - typically $T(n) = O(n \log n)$ - for solving Hankel and Toeplitz linear systems - to be used to solve general symmetric linear systems, e.g. found in regression problems. Symmetric linear systems typically have $T(n) = O(n^3)$ (with Conjugate Gradient method) or at best $T(n) = O(n^{2+\alpha}); \alpha \in < 0, 1 >$ (with Strassen's method).

Keywords: Hankel, Toeplitz, Matrix Series Expansion

1 Introduction

Definition 1 (Partial-Hankel). *A partial-Hankel matrix ($n\,x\,n$) is a symmetric matrix with a Hankel matrix ($k\,x\,k$; $k \leq n$) with a boundary of 0's of size $\frac{n-k}{2}$ centered around the Hankel matrix. If $k = n$ a partial-Hankel matrix is an ordinary Hankel matrix.*

Definition 2 (Peel Matrix). *An peel matrix ($n \times n$) is a matrix with the outermost non-zero region set to zero. This is performed by the peel() function, e.g.*

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{12} & b_{22} & b_{23} & b_{24} \\ b_{13} & b_{23} & b_{33} & b_{34} \\ b_{14} & b_{24} & b_{34} & b_{44} \end{pmatrix} \quad (1)$$

$$peel(B) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & b_{22} & b_{23} & 0 \\ 0 & b_{23} & b_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (2)$$

Theorem 1. *A symmetric matrix B ($n\,x\,n$) is equivalent to a series of $\frac{n}{2}$ partial-Hankel matrices.*

Proof. (sketch)

Let B be a 4×4 symmetric matrix

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{12} & b_{22} & b_{23} & b_{24} \\ b_{13} & b_{23} & b_{33} & b_{34} \\ b_{14} & b_{24} & b_{34} & b_{44} \end{pmatrix} \tag{3}$$

Let C be a 4×4 Hankel matrix generated by the exterior rows and columns of B.

$$C = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{12} & b_{13} & b_{14} & b_{24} \\ b_{13} & b_{14} & b_{24} & b_{34} \\ b_{14} & b_{24} & b_{34} & b_{44} \end{pmatrix} \tag{4}$$

Let D=$peel(C)$

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & b_{13} & b_{14} & 0 \\ 0 & b_{14} & b_{24} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5}$$

and E=$peel(B)$

$$E = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & b_{22} & b_{23} & 0 \\ 0 & b_{23} & b_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{6}$$

Then

$$partial - Hankel - series(B) = C - peel(C) + peel(B) = C - D + E \tag{7}$$

In the general case with $n \times n$ matrix B : E is symmetric, so we get a recursive peeling property holds and we get B is a sum of partial-Hankel matrices, each of the matrices in the sum become increasingly sparse (but dense and Hankel-type around the middle).

general recursive case

$$partialHankelSeries(B) = exteriorHankelize(B) - \tag{8}$$
$$peel(exteriorHankelize(B)) + \tag{9}$$
$$partialHankelSeries(peel(B)) \tag{10}$$

Conjecture 1 (Approximation). B may be approximated by adding together $t <= n$ partial-Hankel matrices. If $t = n$ the sum is equivalent to B.

2 Open Questions

1. Is this a valid and well-known transformation?
2. How to determine the approximation accuracy as a function of t?
3. Can this approximation be used sequentially to efficiently solve symmetric linear systems?
4. Can this approximation be used in parallel to efficiently solve symmetric linear systems?